Adjunctions on the lattice of hierarchies

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Abstract

Hierarchical segmentation produces not a fixed partition but a series of nested partitions, also called hierarchy. The structure of a hierarchy is univocally expressed by an ultrametric 1/2-distance. The lattice structure of hierarchies is equivalent with the lattice structure of their ultrametric 1/2-distances.

The hierarchies form a complete sup- and inf- generated lattice on which an adjunction can be defined.

1 Introduction

Hierarchies are the classical structure for representing a taxinomy. The most famous taxonomy, the Linnaean system classified nature within a nested hierarchy, starting with three kingdoms. Kingdoms were divided into Classes and they, in turn, into Orders, which were divided into Genera (singular: genus), which were divided into Species (singular: species). Below the rank of species he sometimes recognized taxa of a lower (unnamed) rank (for plants these are now called "varieties").

Hierarchies are also useful in the domain of image processing, as they represent in a condensed way nested partitions obtained through image segmentation. Hierarchies appear quite naturally in the field of morphological segmentation, which uses as tool the watershed of gradient images. As a matter of fact, the catchment basins of a topographic surface form a partition. If a basin is flooded and does not contain a regional minimum anymore, it is absorbed by a neighboring basin and vanishes from the segmentation. A hierarchy is hence obtained by considering the catchment basins associated to increasing degrees of flooding. They structure the information on the image by weighting the importance of the contours : the importance of a contour being measured by the level of the hierarchy where it disappears [3].

After an axiomatic definition of a hierarchy, we show that hierarchies are characterized by an ultrametric ecart (also called ultrametric half-distance [5]). This ecart permits quite simply to define and analyze the complete lattice structure of hierarchies. They also permit to define two adjunctions between hierarchies. Some examples of hierarchies met in morphological segmentation are given.

Hierarchies appear as natural generalizations of partitions. A hierarchy is characterized by an ultrametric half-distance, whereas a partition is characterized by an ultrametric binary half-distance. The algebraic structure of partitions has been studies by Serra, Heijmans, Ronse ([2],[8],[4]). The same set may be partitioned into distinct partitions according the type of connectivity one adopts. Serra has laid down the adequate framework for extending the topological notion of connectivity by defining connective classes ([6], [8]). We extend this construction to hierarchies by introducing taxonomy classes.

2 Definition of hierarchies

2.1 Axiomatic definition of hierarchies

The axiomatic definition of hierarchies is due to Benzecri [1].

2.1.1 Definition of a partition

Let E be a domain whose elements are called points. Let Π be a subset of $\mathcal{P}(E)$. Π is a partition of E if the following conditions are verified:

- $\bigcup\limits_{B \in \diamond} B = E$
- $B_1, B_2 \in \Pi$ imply $B_1 \cap B_2 = \varnothing$ or $B_1 = B_2$

2.1.2 Definition of a dendrogram and its elements in $\mathcal{P}(E)$

Let $\mathcal{X}$ be a subset of $\mathcal{P}(E)$, on which we consider the inclusion order relation. $\mathcal{X}$ is a dendrogram if the following axiom is verified :

Axiom 1 ***(Dendrogram axiom)*** $A, U, V \in \mathcal{X}$:
$A \subset U$ *and* $A \subset V \Rightarrow U \subset V$ *or* $V \subset U$

An example : if the tiger is simultaneously a mammal and an animal, then any mammal is an animal or any animal is a mammal. A, U and V belong to three different levels of the hierarchy. Partitions on the contrary consider only one level, and distinct elements have an empty intersection. As a matter of fact, the dendrogram axiom is weaker than the axiom defining partitions. if U, V are classes of a partition and there exists a set A included in both U and V then $U = V$.

If $\mathcal{X}$ is a dendrogram, we may define :
- the summits : $\mathrm{Sum}(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : A \subset B \Rightarrow A = B\}$
- the leaves : $\mathrm{Leav}(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : B \subset A \Rightarrow A = B\}$
- the nodes : $\mathrm{Nod}(\mathcal{X}) = \mathcal{X} - \mathrm{Leav}(\mathcal{X})$

$\mathcal{X}$ is a hierarchy, if the two following axioms are verified:

Axiom 2 ***(Intersection axiom) :*** *two elements of $\mathcal{X}$ which are not comparable for the inclusion order have an empty intersection:* $A, B \in \mathcal{X}$ *:* $A \cap B \in \{A, B, \emptyset\}$

Axiom 3 *(**Union axiom**) Any element A of $\mathcal{X}$ is the union of all other elements of $\mathcal{X}$ contained in A:*
$\forall A \in \mathcal{X} : \bigcup \{B \in \mathcal{X} \mid B \subset A \,; B \neq A\} = \{A, \emptyset\}$

Proposition 4 *The intersection axiom implies that A is a dendrogram for the inclusion order.*

Proof. If $A \neq \emptyset$, $A \subset U$ and $A \subset V$, then $U \cap V \neq \emptyset$, implying that $U \cap V = U$ or $U \cap V = V$, that is $V \subset U$ or $U \subset V$ showing that the dendrogram axiom is satisfied. ■

A series of partitions $(\Pi_i)_{i \in I}$ is said to be nested, if each region of Π_j is the union of regions of Π_i for $j > i$. Considering all tiles belonging to a series of nested partitions $(\Pi_i)_{i \in I}$, obviously yields a hierarchy $\mathcal{X}$.

2.1.3 Stratified hierarchies, ultrametric distances and nested partitions

$\mathcal{X}$ is a stratified hierarchy, if it is equipped with an index function st from $\mathcal{X}$ into the interval $[0, L]$ of $\mathbb{R}$ which is strictly increasing with the inclusion order:
$\forall A, B \in \mathcal{X} : A \subset B$ and $B \neq A \Rightarrow \mathrm{st}(A) < \mathrm{st}(B)$.
Stratification offers the possibility of thresholding a hierarchy: the elements A of a hierarchy verifying $\mathrm{st}(A) \geq \lambda$ are all coarser than λ.

Given a stratified hierarchy $\mathcal{X}$, verifying $\mathrm{st}(A) = 0$ for each $A \in \mathrm{Leav}(\mathcal{X})$, a distance between the elements of $\mathcal{P}(E)$ is defined by:
$\forall C, D \in \mathcal{P}(E)$, $d(C, D) = \inf \{\mathrm{st}(A) \mid A \in \mathcal{X} : C \subset A \text{ and } D \subset A\}$.
Properties : d is an ultrametric distance :
$\forall A, B \in \mathcal{X} \quad d(A, B) = 0 \Rightarrow A = B$
$\forall C, D \in \mathcal{P}(E) \quad d(C, D) = d(D, C)$
$\forall B, C, D \in \mathcal{P}(E) \quad d(C, D) \leq \max \{d(C, B), d(B, D)\}$

This last inequality is called ultrametric inequality, it is stronger than the triangular inequality. It expresses that the index of the smallest set containing C and D is smaller or equal than the index or the smallest set containing all three elements B, C and D, and whose diameter is $\max \{d(C, B), d(B, D)\}$.

For $X \in \mathcal{P}(E)$ the closed ball of centre X and radius ρ is defined by $\mathrm{Ball}(X, \rho) = \{D \in \mathcal{P}(E) \mid d(X, D) \leq \rho\}$.

2.1.4 Hierarchies as balls of an ultrametric distance

Inversely, given an ultrametric distance index χ, the closed balls of radius λ form a partition. For increasing values of λ these partitions are nested and become coarser and coarser. Hence we obtain like that a stratified hierarchy. In order to establish it, we prove the three following lemmas.

Lemma 5 *Two closed balls* $\mathrm{Ball}(X, \rho)$ *and* $\mathrm{Ball}(Y, \rho)$ *with the same radius are either disjoint or identical.*

Proof. Consider two closed balls $\mathrm{Ball}(X, \rho)$ and $\mathrm{Ball}(Y, \rho)$ with a non empty intersection and let A be an element in this intersection. Then necessarily $\mathrm{Ball}(X, \rho) = \mathrm{Ball}(Y, \rho)$. Let us show for instance the inclusion $\mathrm{Ball}(X, \rho) \subset \mathrm{Ball}(Y, \rho)$. Let $B \in \mathrm{Ball}(X, \rho)$, then $\chi(Y, B) \leq \chi(Y, A) \vee \chi(A, X) \vee \chi(X, B) \leq \rho$, showing that $B \in \mathrm{Ball}(Y, \rho)$ ■

Lemma 6 *Each element of a closed ball* $\mathrm{Ball}(X, \rho)$ *is centre of this ball*

Proof. Suppose that B is an element of $\mathrm{Ball}(A, \rho)$. Let us show that then B also is centre of this ball. $\mathrm{Ball}(B, \rho)$ and $\mathrm{Ball}(A, \rho)$ have the element B in common, hence, according to the preceding lemma they are identical. ■

Lemma 7 *The radius of a ball is equal to its diameter.*

Proof. Let $\mathrm{Ball}(A, \rho)$ be a ball of diameter λ, that is the maximal distance between two elements of the ball. Hence $\rho \leq \lambda$. Let B and C be two extremities of a diameter in $\mathrm{Ball}(A, \rho)$: $\lambda = \chi(B, C) \leq \chi(B, A) \vee \chi(A, C) = \rho$. Hence $\lambda = \rho$. ■

Since two closed balls of same radius ρ are either identical or disjoint, they form a partition. For increasing values of ρ , the balls are also increasing, hence we obtain nested partitions.

For this reason, given an ultrametric distance d, the closed balls of radius λ form a partition. For increasing values of λ, these partitions are nested, become coarser and coarser and form a stratified hierarchy.

Remark 8 *Instead of closed balls, we could have taken open balls. The results are the same.*

Partitions as particular hierarchies Partitions are hierarchies with only two stratification levels. For all elements A of a partition Π, we have $\mathrm{st}(A) = 0$. The only element for which we have $\mathrm{st}(B) = 1$ is the domain E itself. The associated ultrametric distance π also is binary. For two distinct tiles of a partition B_1, B_2, we have $\pi(B_1, B_2) = 1$, whereas $\pi(B_1, B_1) = 0$

2.2 Extending the hierarchies to the points of E

2.2.1 Hierarchies

Consider a stratified hierarchy $\mathcal{X}$ to which is associated an ultrametric distance χ. The hierarchy $\mathcal{X}$ is a collection of subsets of $\mathcal{P}(E)$, each of them containing points of E. We designate with the letters $p, q, r...$ the points of E. If E belongs to the hierarchy, then each point p of E is contained in a unique leave $A = \mathrm{Leav}(\mathcal{X})$ of $\mathcal{X}$.. As this leave is unique we call it $\mathrm{Leav}(p)$.

We now extend the ultrametric distance on $\mathcal{X}$ into an ultrametric ecart on the points of E : for $p, q \in E$: $\chi(p, q) = \chi(\mathrm{Leav}(p), \mathrm{Leav}(q))$

Its properties are :

- for $p \in E$: $\chi(p, p) = \chi(\mathrm{Leav}(p), \mathrm{Leav}(p)) = 0$: reflexivity

- for $p, q \in E$: $\text{Leav}(p) = \text{Leav}(q) \Rightarrow \chi(p, q) = 0$
- for $p, q \in E$: $\chi(p, q) = \chi(\text{Leav}(p), \text{Leav}(q)) = \chi(\text{Leav}(q), \text{Leav}(p)) = \chi(q, p)$: symmetry
- for $p, q, r \in E$: $\chi(p, r) = \chi(\text{Leav}(p), \text{Leav}(r)) \leq \chi(\text{Leav}(p), \text{Leav}(q)) \vee \chi(\text{Leav}(q), \text{Leav}(r)) = \chi(p, q) \vee \chi(q, r)$: ultrametric inequality

Hence χ is an ultrametric ecart but not a distance, as the antisymmetry is not verified, since distinct points p and q belonging to a same leave have an ecart equal to 0. On the other hand, the ultrametric inequality is stronger than the triangular inequality characterizing metric distances and ecarts. Laurent Schwartz called these ecarts half distances in [5]. His definition of half-distances and half-metric spaces is given below.

2.2.2 Half distances and half metric spaces

Definition 9 *A half-distance on a domain E is a mapping d from $E \times E$ into $\mathbb{R}^+$ with the following properties:*
1) Symmetry : $d(x, y) = d(y, x)$
2) Half-positivity: $d(x, y) \geq 0$ and $d(x, x) = 0$
3) Triangular inequality: $d(x, z) \leq d(x, y) + d(y, z)$

Definition 10 *A half metric space is a set E with a family $(d_i)_{i \in I}$ of half-distances verifying the following condition:*
the family $(d_i)_{i \in I}$ is a "filtering family", i.e. for any finite subset J of I, there exists an index $k \in I$ such that $d_k \geq d_j$ for all $j \in J$

The open half balls $B_{i,o}(a, R)$ (resp. closed $B_i(a, R)$ of a center $a \in E$, of radius R and index i are all x of E such that $d_i(a, x) < R$ (resp. $\leq R$).

A half metric space is then a topological space defined as follows : a subset $\mathcal{O}$ of E is open, if for each point $x \in \mathcal{O}$, there exists a half ball $B_i(x, R)$ centered at x, with a positive radius entirely contained in $\mathcal{O}$.

If the triangular inequality is replaced by the ultrametric inequality, we call it ultrametric half-distance or ultrametric ecart. To any hierarchy $\mathcal{X}$ to defined on subsets of $\mathcal{P}(E)$ is thus associated an ultrametric half-distance or ultrametric ecart.

2.2.3 Equivalence relations and partitions

Consider now the case of a binary ultrametric ecart π defined on the points of E. This ecart permits to define the binary relation R:

$$\text{For } p, q \in E : \; p \; R \; q \Leftrightarrow \pi(p, q) = 0$$

This relation is reflexive and symmetrical as is π itself.

Consider now $p, q, r \in E$ verifying $p \; R \; q$ and $q \; R \; r$. This is equivalent with $\pi(p, q) = 0$ and $\pi(q, r) = 0$. But then $\pi(p, r) \leq \pi(p, q) \vee \pi(q, r) = 0$ indicating that $p \ R \ r$. Hence the relation R is also transitive: it is an equivalence relation. The classes of this equivalence relation precisely are the leaves of the ecart π and form a partition of E.

2.2.4 Partial partitions and partial hierarchies

Consider a subset A of $\mathcal{P}(E)$. A hierarchy $\mathcal{X}$ (resp. a partition Π) on A is called partial hierarchy (resp. partial partition) on E. The domain A is called support of the partial hierarchy (resp. partial partition) and is written $\mathrm{supp}(\mathcal{X})$ (resp. $\mathrm{supp}(\Pi)$)

The notion of partial partition has been introduced and studied by Ch. Ronse in [4] ; Ronse denotes $\Pi^*(E)$ the class of partial partitions of E. Likewise let us denote $\mathcal{X}^*(E)$ the class of partial hierarchies.

Partial partitions Consider a partial partition Π^* in $\Pi^*(E)$. Π^* is a normal partition on $\mathrm{supp}(\Pi^*)$. So the binary ultrametric ecart $\pi^*(p, q)$ is perfectly defined for $p, q \in \mathrm{supp}(\Pi^*)$. In order to completely characterize the partial partition without the need to specify its support, we extend π^* in order to get π defined on $E \times E$:

- (pp1) : for $p, q \in \mathrm{supp}(\Pi^*) : \pi(p, q) = \pi^*(p, q)$ is an ultrametric ecart
- (pp2) for $p \notin \mathrm{supp}(\Pi^*), \forall q \in E : \pi(p, q) = 1$

This last relation is also true for p itself : for $p \notin \mathrm{supp}(\Pi^*) : \pi(p, p) = 1$

The support $\mathrm{supp}(\Pi^*)$ is characterized by $\{p \mid \pi(p, p) = 0\}$.

We call $\mathrm{cl}(p)$ the closed ball of centre p and of radius 0 associated to π. Relation (pp2) implies that for $p \notin \mathrm{supp}(\Pi^*)$ the class $\mathrm{cl}(p)$ is empty.

Consider now $p, q \in E$ such that $q \in \mathrm{cl}(p)$. This shows that $p, q \in \mathrm{supp}(\Pi^*)$ and $\pi^*(p, q) = 0$. If $r \in \mathrm{cl}(p)$; then $\pi^*(p, r) = 0$ and $\pi^*(q, r) \leq \pi^*(q, p) \vee \pi^*(p, r) = 0$ showing that $r \in \mathrm{cl}(q)$. Similarly $r \in \mathrm{cl}(q) \Rightarrow r \in \mathrm{cl}(p)$.

Hence for any $p, q \in E$, $q \in \mathrm{cl}(p) \Rightarrow \mathrm{cl}(q) = \mathrm{cl}(p)$.

But these are precisely the criteria given by Ronse for defining partial partitions:

- (P1b) for any $p \in E$, $\mathrm{cl}(p) = \varnothing$ or $p \in \mathrm{cl}(p)$
- (P2a) for any $p, q \in E$, $q \in \mathrm{cl}(p) \Rightarrow \mathrm{cl}(q) = \mathrm{cl}(p)$

As shown by Ronse, to partial partitions correspond partial equivalence relations, which are symmetric and transitive but are not reflexive. The support of a partial equivalence relations R being the set of all points $p \in E$ for which there exists a point $q \in E$ verifying $p \ R \ q$.

Partial hierarchies Consider a subset A of $\mathcal{P}(E)$. A hierarchy $\mathcal{X}^*$ on a subset A of $\mathcal{P}(E)$ is called partial hierarchy E. The domain A is called support of the partial hierarchy (resp. partial partition) and is written $\text{supp}(\mathcal{X}^*)$. We call $\mathcal{X}^*(E)$ the family of partial hierarchies on E.

Consider a partial hierarchy $\mathcal{X}^*$ represented by its ultrametric ecart χ^*. The ecart χ^* can then be extended to an ultrametric mapping χ:

- (xp1) : for $p, q \in \text{supp}(\mathcal{X}^*) : \chi(p,q) = \chi^*(p,q)$ is the ultrametric ecart on A
- (xp2) for $p \notin \text{supp}(\mathcal{X}^*), \forall q \in E : \chi(p,q) = L$

This last relation is also true for p itself : for $p \notin \text{supp}(\Pi^*) : \chi(p,p) = \lambda$
The support $\text{supp}(\Pi^*)$ is characterized by $\{p \mid \chi(p,p) = 0\}$.

Aliens and singletons Partial hierarchies and partial partitions have thus two domains, the support domain and its complementary set.

We call "aliens" the points verifying $\forall q \in E : \pi(p,q) > 0$, that is the points outside the support.

Aliens should not be mixed up with the singletons, which duly belong to the support. The singleton $\{x\}$ is a set of $\mathcal{P}(E)$ reduced to the point x. Singletons are characterized by: $\forall q \in E\ ,\ p \neq q, : \pi(p,q) \geq 0$ and $\pi(p,p) = 0$.

Transforming a set into a partition Given a set A of $\mathcal{P}(E)$, we have three ways to complete it, in order to create a partition:

- $\text{Backg}(A) = A \cup \overline{A}$: the set A and its complement $\overline{A}$ form a partition
- $\text{singl}(A)$ is the partition where $\overline{A}$ is pulverized into singletons
- $\text{alien}(A)$ is the partition where $\overline{A}$ is pulverized into aliens

2.2.5 Dilations and erosions associated to partitions and hierarchies

An adjunction associated to a partition Π To any partition Π on E we may associate a dilation δ. For a point $p \in E$, one defines $\delta(p) = \text{cl}(p)$. One then defines $\delta(X) = \bigcup \{\delta(x) \mid x \in X\} = \bigcup \{C_i \in \Pi \mid X \cap C_i \neq \emptyset\}$

The properties of δ are the following :

- δ is increasing and commutes with union : it is indeed a dilation
- obvioulsy $x \in \delta(x)$, hence δ is extensive
- it is also a closing. The fact that δ also is a closing seems at first sight strange, as the class of invariants of a closing is stable by intersection. But the invariants of δ are unions of classes of the partition Π. Hence their class is stable by intersection. It is easy to check that δ is a dilation-closing : $\delta\delta(Y) = \delta(Y) \Rightarrow \delta(Y) \subset \varepsilon\delta(Y)$ by adjunction ; but ε being anti-extensive, we have $\delta(Y) \subset \varepsilon\delta(Y) \subset \delta(Y)$, hence $\delta(Y) = \varepsilon\delta(Y)$. By duality, we have $\varepsilon = \delta\varepsilon$.

Let us now study the erosion ε adjunct to δ : $Y \subset \varepsilon(X) \Leftrightarrow \delta(Y) \subset X$.

Obviously $\varepsilon(X) = \bigcup\{Y \mid Y \subset \varepsilon(X)\} = \bigcup\{Y \mid \delta(Y) \subset X\}$. But since δ is extensive and idempotent
$\bigcup\{Y \mid \delta(Y) \subset X\} = \bigcup\{\delta(Y) \mid \delta(Y) \subset X\} = \bigcup\{C_i \in \Pi \mid C_i \subset X\}$.

By duality ε is increasing, anti-extensive, idempotent and commutes with intersection, it is an erosion-opening: $\varepsilon = \delta\varepsilon$

Adjunctions associated to a hierarchy $\mathcal{X}$. The closed balls $\mathrm{Ball}(p, \rho)$ of radius ρ form a partition, for which we may apply the results of the previous paragraph and define the adjunction $(\delta_\lambda, \varepsilon_\lambda)$ defined by:

- $\delta_\rho(X) = \bigcup\{\mathrm{Ball}(x, \rho) \mid x \in X\}$
- $\varepsilon_\rho(X) = \bigcup\{Y \mid Y \subset \varepsilon(X)\} = \bigcup\{Y \mid \delta(Y) \subset X\}$
  $= \bigcup\{\mathrm{Ball}(x, \rho) \mid x \in E\ ,\ \mathrm{Ball}(x, \rho) \subset X\}$

Applications : interactive segmentation The following examples have been developed within a toolbox for interactive segmentation ([9]).

Intelligent brush An intelligent brush segments an image by "painting" it: it first selects a zone of interest by painting. Contrary to conventional brushes, the brush adapts its shape to the contours of the image. The shape of the brush is given by the region of the hierarchy containing the cursor. Moving from one place to another changes the shape of the brush, when one goes from one tile of a partition to its neighboring tile. Going up and down the hierarchy modifies the shape of the brush. In fig.1, on the left, one shows the trajectory of the brush ; in the centre, the result of a fixed size brush, and on the right the result of a self-adapting brush following the hierarchy. This self adapting brush is nothing by the dilation δ_r by a ball associated to the hierarchy, centered at the position of the mouse and of a radius, also easily modified through the mouse. This method has been used with success in a package for interactive segmention of organs in 3D medical images.

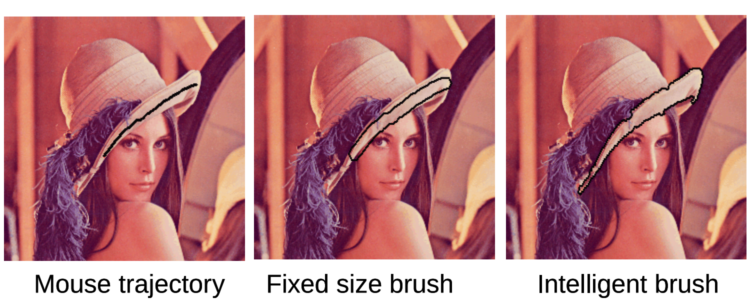


Figure 1: Comparison of the drawing with a fixed size brush and a self adaptive brush.

Figure 2: Left: initial image
Center: result of the magic wand
Right ; smallest region of the hierarchy containing the magic wand.

Magic wand The magic wand in a conventional computer graphics toolbox consists in extracting the region which touches the position of the mouse and whose colour lies within some predefined limits from the coulour at the mouse position. The next step consists in replacing this set by the smallest set of the hierarchy which contains it. This operation is a closing, described by Ch. Ronse in [4]. The result is shown in fig.2

3 The lattice of hierarchies

It is often interesting to combine several hierarchies, in order to combine various criteria or merge the information obtained from diverse sources (colour or multispectral images for instance). We first define an order relation between hierarchies which structures them into a complete lattice.

3.0.6 Order relation

Complete hierarchies Let $\mathcal{A}$ and $\mathcal{B}$ be two stratified hierarchies, with their associated half-distances : $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$. The following relation defines an order relation between the hierarchies: $B < A \Leftrightarrow \forall p, q \in E \quad \chi_{\mathcal{A}}(p,q) \leq \chi_{\mathcal{B}}(p,q)$

It follows that $\forall p \in E : \text{Ball}_{\mathcal{B}}(p, \rho) \subset \text{Ball}_{\mathcal{A}}(p, \rho)$

With this order relation the hierarchies of $\mathcal{P}(E)$ form a complete lattice. The maximal element is the hierarchy having E as only element and the smallest hierarchy contains only singletons $\{x\}$.

Partial hierarchies This order also holds for partial hierarchies. Let $\mathcal{A}$ and $\mathcal{B}$ be two stratified partial hierarchies, with their associated half-distances : $\chi^*_{\mathcal{A}}$ and $\chi^*_{\mathcal{B}}$. The following relation defines an order relation between the hierarchies: $B < A \Leftrightarrow \forall p, q \in E \quad \chi^*_{\mathcal{A}}(p,q) \leq \chi^*_{\mathcal{B}}(p,q)$. For each $p \notin \text{supp}(A) : \chi^*_{\mathcal{A}}(p,p) =$

L, which implies that $\chi^*_{\mathcal{B}}(p,p) = L$, indicating that $\overline{\text{supp}(\mathcal{A})} \subset \overline{\text{supp}(\mathcal{B})}$, or equivalently $\text{supp}(\mathcal{B}) \subset \text{supp}(\mathcal{A})$

The smallest partial hierarchy contains only aliens, i.e. points p verifying $\forall q \in E\ ,\ \chi(p,q) = L$.

Particular case of partitions Let $\mathcal{A}$ and $\mathcal{B}$ be two partitions, with their associated binary half-distances : $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$. The partition $\mathcal{B}$ is finer than the partition $\mathcal{A}$ iff $\forall p,q \in E \quad d_{\mathcal{A}}(p,q) \leq d_{\mathcal{B}}(p,q)$

It follows that $\forall p \in E : \text{Ball}_{\mathcal{B}}(p,\rho) \subset \text{Ball}_{\mathcal{A}}(p,\rho)$

But the balls of a partition $\text{Ball}_{\mathcal{B}}(p,\rho)$ are the tiles of this partition. Hence the tiles of the finer partition $\mathcal{B}$ are included in the tiles of the coarser partition $\mathcal{A}$ which is coherent with the usual definition of the order between partitions.

3.0.7 Infimum of two hierarchies

Complete hierarchies The infimum of two hierarchies $\mathcal{A}$ and $\mathcal{B}$ is written $\mathcal{A} \wedge \mathcal{B}$ and is defined by its ultrametric half-distance $d_{\mathcal{A}\wedge\mathcal{B}} = d_{\mathcal{A}} \vee d_{\mathcal{B}}$. It is easy to check that it is indeed a half-distance. It is symmtrical and half-positive. Let us check the ultrametric inequality:

$(d_{\mathcal{A}} \vee d_{\mathcal{B}})(p,r) \vee (d_{\mathcal{A}} \vee d_{\mathcal{B}})(r,q) = (d_{\mathcal{A}}(p,r) \vee d_{\mathcal{A}}(r,q)) \vee (d_{\mathcal{B}}(p,r) \vee d_{\mathcal{B}}(r,q)) > (d_{\mathcal{A}}(p,q) \vee d_{\mathcal{B}}(p,q)) = d_{\mathcal{A}} \vee d_{\mathcal{B}}(p,q)$

Its balls are defined by : $\forall p \in E : \text{Ball}_{\mathcal{A}\wedge\mathcal{B}}(p,\rho) = \text{Ball}_{\mathcal{A}}(p,\rho) \wedge \text{Ball}_{\mathcal{B}}(p,\rho)$

Partial hierarchies If $\mathcal{A}$ and $\mathcal{B}$ are partial hierarchies, their supremum is defined as for the hierarchies. The aliens of a partial hierarchy $\mathcal{X}$ are characterized by $\forall p,q \in E : \chi(p,q) = L$. Hence the aliens of $\mathcal{A} \wedge \mathcal{B}$ are the union of the aliens of $\mathcal{A}$ and of $\mathcal{B}$, i.e. $\overline{\text{supp}(\mathcal{A} \wedge \mathcal{B})} = \overline{\text{supp}(\mathcal{A})} \vee \overline{\text{supp}(\mathcal{B})}$ or equivalently $\text{supp}(\mathcal{A} \wedge \mathcal{B}) = \text{supp}(\mathcal{A}) \wedge \text{supp}(\mathcal{B})$.

3.0.8 Infimum of two hierarchies

The subdominant ultrametric half-distance The supremum of two hierarchies $\mathcal{A}$ and $\mathcal{B}$ is written $\mathcal{A} \vee \mathcal{B}$ and is the smallest hierarchy larger than $\mathcal{A}$ and $\mathcal{B}$.

As $d_{\mathcal{A}} \wedge d_{\mathcal{B}}$ is not an ultrametric distance, we chose for $d_{\mathcal{A}\vee\mathcal{B}}$ the largest ultrametric distance which is lower than $d_{\mathcal{A}} \wedge d_{\mathcal{B}}$. This distance exists: the set of ultrametric distances lower than $d_{\mathcal{A}} \wedge d_{\mathcal{B}}$ is not empty, as the distance 0 is ultrametric ; furthermore, this family is closed by supremum, hence it has a largest element. Let us construct it.

Consider a series of points $(x_0, x_1, \cdots, x_n)$. As $d_{\mathcal{A}\vee\mathcal{B}}$ should be an ultrametric distance, we have for any path $x_0, x_1, ..., x_n$
$d_{\mathcal{A}\vee\mathcal{B}}((x_0,x_n) \leq d_{\mathcal{A}\vee\mathcal{B}}(x_0,x_1) \vee d_{\mathcal{A}\vee\mathcal{B}}(x_1,x_2) \vee \cdots \vee d_{\mathcal{A}\vee\mathcal{B}}(x_{n-1},x_n)$.
But for each pair of points x_i, x_{i+1} we have $d_{\mathcal{A}\vee\mathcal{B}}(x_i,x_{i+1}) \leq [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_i,x_{i+1})$.
Hence $d_{\mathcal{A}\vee\mathcal{B}}(x_0,x_n) \leq [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_0,x_1) \vee [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_1,x_2) \vee \cdots \vee d\, [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_{n-1},x_n)$.
There exists a chain along which the expression on the right becomes minimal and is equal to the maximal value taken by $[d_{\mathcal{A}} \wedge d_{\mathcal{B}}]$ on two successive points of

the chain. This maximal value is called sup section of the chain for $d_{\mathcal{A}} \wedge d_{\mathcal{B}}$. For this reason, the chain itself is called chain of minimal sup-section. This valuation being an ultrametric ecart necessarily is the largest ultrametric ecart below $d_{\mathcal{A}} \wedge d_{\mathcal{B}}$. Let us verify the ultrametric inequality.

For $p, q, r \in E$ there exists a chain between p and q along which $[d_{\mathcal{A}} \wedge d_{\mathcal{B}}](p,q)$ takes its value and another chain between q and r along which $[d_{\mathcal{A}} \wedge d_{\mathcal{B}}](q,r)$ takes its value. The concatenation of both chains forms a chain between p and q which is not necessarily the chain of lowest sup-section between them, hence: $[d_{\mathcal{A}} \wedge d_{\mathcal{B}}](p,r) \leq [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](p,q) \vee [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](q,r)$.

We write $\widetilde{d_{\mathcal{A}} \wedge d_{\mathcal{B}}}$ for the subdominant ultrametric associated to $d_{\mathcal{A}} \wedge d_{\mathcal{B}}$.

Partial hierarchies If $\mathcal{A}$ and $\mathcal{B}$ are partial hierarchies, their infimum is defined as for the hierarchies. The aliens of a partial hierarchy $\mathcal{X}$ are characterized by $\forall p, q \in E : \chi(p,q) = L$. Hence the chains characterizing the subdominant ultrametric distance associated to $\mathcal{A} \vee \mathcal{B}$ avoid the supports $\text{supp}(\mathcal{A})$ and $\text{supp}(\mathcal{B})$.

Geometrical interpretation Suppose that $(x_0, x_1, \cdots, x_n)$ is the chain for which $\widetilde{d_{\mathcal{A}} \wedge d_{\mathcal{B}}}(x_0, x_n) = [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_0, x_1) \vee [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_1, x_2) \vee \cdots \vee [d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_{n-1}, x_n)$ is minimal with a value λ. Then $[d_{\mathcal{A}} \wedge d_{\mathcal{B}}](x_i, x_{i+1}) \leq \lambda$ means that the ball $\text{Ball}_{\mathcal{A}}(x_i, \lambda)$ or the ball $\text{Ball}_{\mathcal{B}}(x_i, \lambda)$ contains the point x_{i+1}. If it is $\text{Ball}_{\mathcal{A}}(x_i, \lambda)$, then x_{i+1} also is center of this ball. Hence a series of points $x_k, x_{k+1}, x_{k+2,\cdots}$ all belong to the same ball $\text{Ball}_{\mathcal{A}}(x_i, \lambda)$, they are all centers of this ball and it is possible to keep only one of them and suppress all others from the list. Like that we get a path where the firsts two points x_0, x_1 belong to one of the balls, say $\text{Ball}_{\mathcal{A}}(x_0, \lambda)$, the couple x_1, x_2 belong to the other $\text{Ball}_{\mathcal{B}}(x_2, \lambda)$, and so on. The successive ovelapping pairs of points belong alternatively to balls $\text{Ball}_{\mathcal{A}}$ or $\text{Ball}_{\mathcal{B}}$.

The necessity of chaining blocks for obtaining suprema of partitions is well known [6] ; Ronse has confirmed that it is still the case for partial partitions [4].

Illustration If $\mathcal{A}_\lambda$, $\mathcal{B}_\lambda$ and $\mathcal{A}_\lambda \vee \mathcal{B}_\lambda$ are the partitions obtained by taking the balls of radius λ in each of the three hierarchies, then the boundaries of $\mathcal{A}_\lambda \vee \mathcal{B}_\lambda$ are all boundaries existing in both $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$. The infimum and supremum of two hierarchies are illustrated in fig.3

3.1 Lexicographic fusion of stratified hierarchies

Let $\mathcal{A}$ and $\mathcal{B}$ be two stratified hierarchies, with their associated distances $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$. In some cases, one of the hierarchies correctly represents the image to segment, but with a too small number of nested partitions. One desires to enrich the current ranking of regions as given by $\mathcal{A}$, by introducing some intermediate levels in the hierarchy. The solution is to combine the hierarchy $\mathcal{A}$ with another hierarchy $\mathcal{B}$ in a lexicographic order.

One produces the lexicographic hierarchy $\text{Lex}(\mathcal{A}, \mathcal{B})$ by defining its ultrametric distance ; it is the largest ultrametric distance below the lexicographic

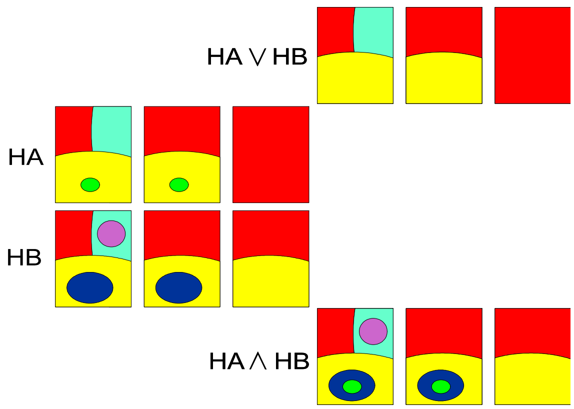


Figure 3: Two hierarchies HA and HB and their derived supremum and infimum

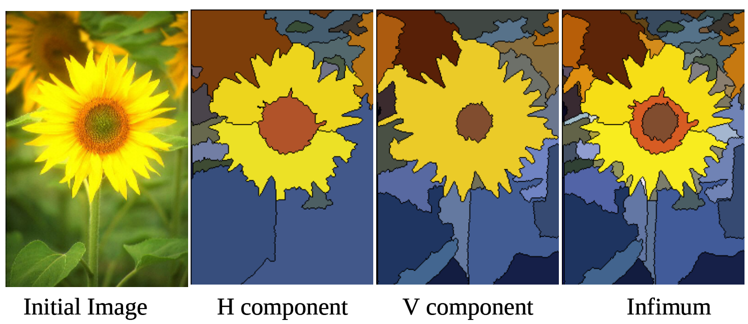


Figure 4: Supremum of two hierarchies.

distance $d_{\mathcal{A},\mathcal{B}}$ classically defined by

$$d_{\mathcal{A},\mathcal{B}}(C,D) > d_{\mathcal{A},\mathcal{B}}(K,L) \Leftrightarrow$$
$$d_{\mathcal{A}}(C,D) > d_{\mathcal{A}}(K,L)$$
$$or$$
$$d_{\mathcal{A}}(C,D) = d_{\mathcal{A}}(K,L) \text{ and } d_{\mathcal{B}}(C,D) > d_{\mathcal{B}}(K,L)$$

Fig.5 present two hierarchies HA and HB and the derived lexicographic hierarchies Lex($\mathcal{A},\mathcal{B}$) and Lex($\mathcal{B},\mathcal{A}$). Fig. shows an image which is difficult to segment as it contains small contrasted objects, the cars and the landscape and road which are much larger and less contrasted. Two separate segmentation have been performed. The first based on the contrast segments the cars ; the second, based on the "volume" (area of the regions multiplied by the contrast) segments the landscape. The hierarchy of both these segmentations has been thresholded so as to show 30 regions. The lexicographic fusion of both segmentations Lex($Depth, Volume$), also thresholded at 30 regions offers a nice composition of both segmentations.

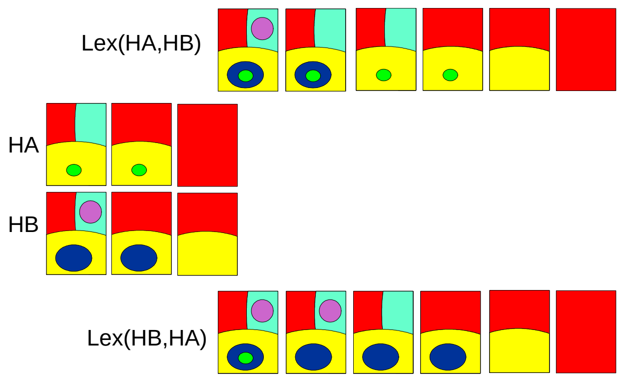


Figure 5: Two hierarchies HA and HB and their derived lexicographic combinations.

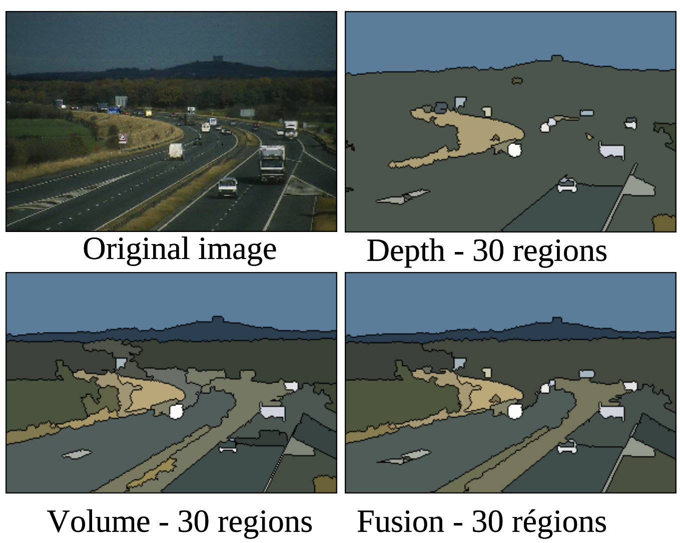


Figure 6: Lexicographic fusion of two hierarchies

4 Connected operators

5 Adjunctions on hierarchies

Given a point O serving as origin, a structuring element B is a family of translations $\bigcup \left\{ \overrightarrow{Ox} \mid x \in B \right\}$. A set X of $\mathcal{P}(E)$ may then be eroded and dilated by this structuring element : the erosion $X \ominus B = \bigwedge\limits_{x \in B} X_{\overrightarrow{Ox}}$ and the dilation $X \oplus B = \bigvee\limits_{x \in B} X_{\overrightarrow{xO}}$. As one uses for one operator the vectors $\overrightarrow{Ox}$ and for the other the vectors $-\overrightarrow{Ox} = \overrightarrow{xO}$, both operators form an adjunction: for any $X, Y \in \mathcal{P}(E)$, we have $X \oplus B < Y \Leftrightarrow X < Y \ominus B$.

A hierarchy $\mathcal{X} \in \mathcal{X}(E)$ is a collection of sets $X^i \in \mathcal{P}(E)$. Through the translation by a vector $\overrightarrow{t}$, these sets $X^i_{\overrightarrow{t}}$ form a new hierarcy $\mathcal{X}_{\overrightarrow{t}}$. If χ is the ultrametric ecart associated to $\mathcal{X}$, the ultrametric ecart associated to $\mathcal{X}_{\overrightarrow{t}}$ will be written $\chi_{\overrightarrow{t}}$.

As the hierarchies form a complete lattice $\mathcal{X}(E)$, we may use the same mechanism for constructing an erosion and a dilation on hierarchies. We define two operators operating on a hierarchy $\mathcal{X}$. For showing that the first $\mathcal{X} \ominus B = \bigwedge\limits_{x \in B} \mathcal{X}_{\overrightarrow{Ox}}$ is an erosion and the second $\mathcal{X} \oplus B = \bigvee\limits_{x \in B} \mathcal{X}_{\overrightarrow{xO}}$ a dilation, we have to show that they form an adjunction.

We have to prove that for any two hierarchies $\mathcal{X}, \mathcal{Y} \in \mathcal{X}(E) : \mathcal{X} \oplus B < \mathcal{Y} \Leftrightarrow \mathcal{X} < \mathcal{Y} \ominus B$.

We will prove the adjunction through the half distance associated to the hierarchies $\mathcal{X}$ and $\mathcal{Y}$.

We have the following correspondances between the hierarchies and the ultrametric ecarts :

- $\mathcal{X} \leftrightarrow \chi$
- $\mathcal{Y} \leftrightarrow \zeta$
- $\mathcal{Y} \ominus B = \bigwedge\limits_{x \in B} \mathcal{Y}_{\overrightarrow{Ox}} \leftrightarrow \bigvee\limits_{x \in B} \zeta_{\overrightarrow{Ox}}$
- $\mathcal{X} \oplus B = \bigvee\limits_{x \in B} \mathcal{X}_{\overrightarrow{xO}} \leftrightarrow \widetilde{\bigwedge\limits_{x \in B} \chi_{\overrightarrow{xO}}}$
- $\mathcal{X} \oplus B < \mathcal{Y} \Leftrightarrow \mathcal{X} < \mathcal{Y} \ominus B \leftrightarrow \widetilde{\bigwedge\limits_{x \in B} \chi_{\overrightarrow{xO}}} > \zeta \Leftrightarrow \chi > \bigvee\limits_{x \in B} \zeta_{\overrightarrow{Ox}}$

Let us now prove the adjunction.

For two arbitrary ultrametric ecarts χ and ζ : $\mathcal{X} < \mathcal{Y} \ominus B \Leftrightarrow \chi > \bigvee\limits_{x \in B} \zeta_{\overrightarrow{Ox}} \Leftrightarrow$ $\forall x \in B : \chi > \zeta_{\overrightarrow{Ox}} \Leftrightarrow \forall x \in B : \chi_{\overrightarrow{xO}} > \zeta \Leftrightarrow \bigwedge\limits_{x \in B} \chi_{\overrightarrow{xO}} > \zeta$

Remains to establish : $\bigwedge\limits_{x \in B} \chi_{\overrightarrow{xO}} > \zeta \Leftrightarrow \widetilde{\bigwedge\limits_{x \in B} \chi_{\overrightarrow{xO}}} > \zeta$:

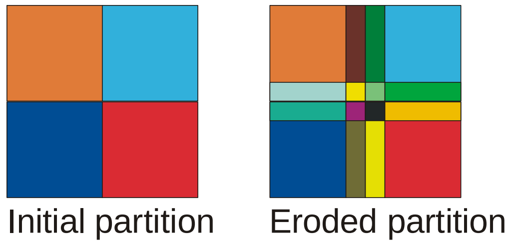


Figure 7: Erosion of a partition

- $\widetilde{\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}} > \zeta \Rightarrow \bigwedge_{x\in B} \chi_{\overrightarrow{xO}} > \zeta$ since $\widetilde{\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}}$ is the largest ultrametric ecart below $\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}$
- Suppose now $\bigwedge_{x\in B} \chi_{\overrightarrow{xO}} > \zeta$. Since ζ is an ultrametric ecart below $\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}$, it is smaller or equal to the largest ultrametric ecart below $\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}$, that is $\widetilde{\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}}$.

This completes the proof :

$\mathcal{X} < \mathcal{Y} \ominus B \Leftrightarrow \chi > \bigvee_{x\in B} \zeta_{\overrightarrow{Ox}} \Leftrightarrow \bigwedge_{x\in B} \chi_{\overrightarrow{xO}} > \zeta \Leftrightarrow \widetilde{\bigwedge_{x\in B} \chi_{\overrightarrow{xO}}} > \zeta \Leftrightarrow \mathcal{X} \oplus B < \mathcal{Y}$

The erosion of a partition by a square structuring element (8 connexity) is illustrated in fig.7

Remark 11 *The adjunction defined for hierarchies is defined in a similar fashion for a partial hierarchy.*

5.1 Decomposition and recomposition of hierarchies

5.1.1 Thresholding

Consider a hierarchy $\mathcal{X}$ with its associated ultrametric ecart χ. By thresholding the ultrametric at level λ one obtains a binary ultrametric ecart :

$$T^{\lambda}(\chi) = \begin{array}{l} 1 \text{ if } \chi > \lambda \\ 0 \text{ if } \chi \leq \lambda \end{array}$$

$T^{\lambda}(\chi)$ characterizes a partition.

The hierarchy can be recovered from its thresholds by $\chi = \bigvee_{\lambda} \lambda T^{\lambda}(\chi)$.

Remark 12 *If $\mathcal{X}$ is a partial hierarchy, then $T^{\lambda}(\chi)$ is a partial partition*

5.1.2 Reconstructing a hierarchy

Consider a stratified hierarchy $\mathcal{X}$.

Inf-generation Consider a hierarchy$\mathcal{X}$, union of a family (A_i) of $\mathcal{P}(E)$. If a set A_i belongs to $\mathcal{X}$, then we construct a partition by adding to A_i its complement. We have defined this operator earlier $\mathrm{Backg}(A_i) = A_i \cup \overline{A_i}$ to which we associate the binary ultrametric ecart α_i.

The ultrametric ecart χ associated to $\mathcal{X}$ is sup-generated and equal to $\bigvee_i \mathrm{st}(A_i) * \alpha_i$ and the hierarchy is inf-generated by the partitions $\mathrm{Backg}(A_i)$ weighted by their stratification level.

This decomposition helps understanding how the erosion of ultrametric hierarchies work. To $\mathcal{X} \ominus B$ is associated $\chi \oplus B = \left[\bigvee_i \mathrm{st}(A_i) * \alpha_i\right] \oplus B = \bigvee_i [\mathrm{st}(A_i) * \alpha_i] \oplus B$

For each set A_i, the partition $A_i \cup \overline{A_i}$ is eroded, that is the new partition $A_i \cup (A_i/A_i \ominus B) \cup \left(\overline{A_i}/\overline{A_i} \ominus B\right) \cup \overline{A_i}$ is created. This partition keeps the same stratification level as A_i itself. This new collection of partitions creates the eroded hierarchy.

Remark 13 *In the particular case of a partition, the sets forming this partition are eroded, and the space left by the erosion is filled by the intersection of all partial partitions* $(A_i/A_i \ominus B) \cup \left(\overline{A_i}/\overline{A_i} \ominus B\right)$*. The result is illustrated by fig.7. This differs from the definition given by J.Serra in [7], where he filled the spaces left empty by singletons.*

Sup-generation Consider a hierarchy $\mathcal{X}$, union of a family (A_i) of $\mathcal{P}(E)$. We associate to the set A_i the following half-distance α_i :

For any $p, q \in A_i : \alpha_i(p,q) = \mathrm{st}(A_i)$. For $p \notin A_i$, and any q we have $\alpha_i(p,q) = L$. We have thus associated to the set A_i a partial partition equal to A_i on A_i, with a stratification level $\mathrm{st}(A_i)$ and containing only aliens on $\overline{A_i}$.

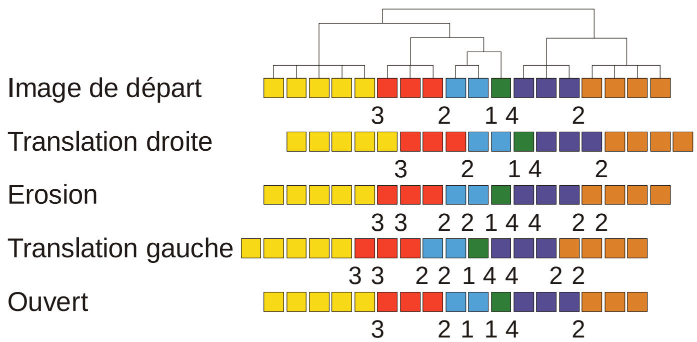
The half-distance χ is then inf-generated and equal to $\bigwedge_i \alpha_i$, corresponding to the sup-generation from the partial hierarchies $\widehat{A_i}$ associated to the α_i.

This decomposition helps understanding how the dilation of ultrametric hierarchies work. To $\mathcal{X} \oplus B = \bigvee_i \widehat{A_i} \oplus B$ is associated the half-distance $\left(\bigwedge_i \alpha_i \ominus B\right)$

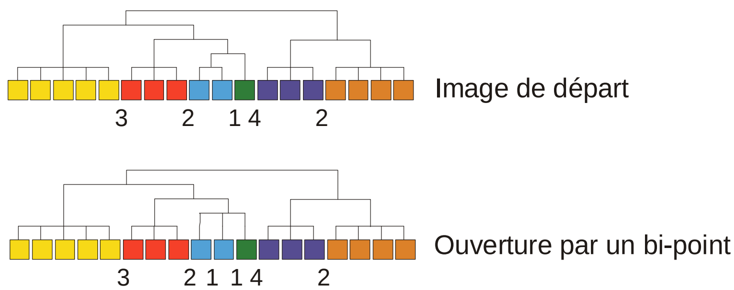
5.2 Illustration

We illustrate the erosion and the opening of a one dimensional hierarchy, first by a structuring element reduced to two pixels, then by a structuring element made of three pixels. In the first case, the erosion and the dilation have to use the structuring element for the erosion and its transposed version for the dilation.

5.2.1 Erosion and opening by a pair of 2 pixels.

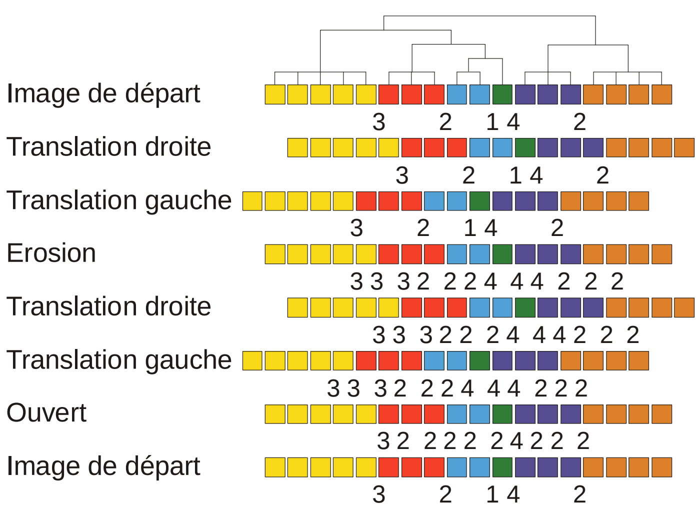


Erosion and opening by a pair of pixels: intermediate steps

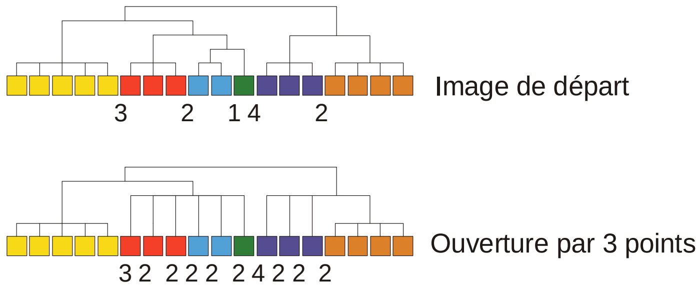


Dendrogram of an initial image and its opening by a segment of 2 points.

5.2.2 Erosion and opening by a centered segment of 3 pixels.



Erosion and opening by a segment of 3 pixels: intermediate steps



Dendrogram of an initial image and its opening by a segment of 3 points.

5.3 Examples of hierarchies

Hierarchies associated to a dissimilarity index A series of nested partitions $(\mathcal{X}_i)$, and hence a hierarchy, may easily be generated from an initial fine partition $\mathcal{X}_0 = \cup R_i,\ i = 1, \ldots, n$ on which a dissimilarity index δ is defined between a subset G of all couples of tiles. For a couple of tiles which do not belong to G, we define a dissimilarity equal to ∞.

If we now take the union of all tiles of $\mathcal{X}_0$ with a dissimilarity index below a given threshold λ, we obtain a coarser partition with a stratification index equal to λ. For increasing values of λ we obtain a series of nested partitions, forming a hierarchy $\mathcal{A}$. The ultrametric distance d associated to this hierarchy is precisely the the subdominant ultrametric distance associated to δ, that is the largest ultrametric distance below δ (see below the supremum of two hierarchies, where the subdominant ultrametric distance also appears) For two tiles A and B of $\mathcal{X}_0$, the subdominant ultrametric distance will be the lowest level λ for which A and B belong to the same tile (if it does not happen, their distance is ∞)

Case of the watershed tesselation If the tessellation is the result of the watershed construction on a gradient image, the dissimilarity measure can be defined as the altitude of the pass point separating two adjacent regions. The ultrametric half distance between two minima is then the "flooding distance" : the flooding distance between two points p and q is the altitude of the lowest flooding for which p and q both belong to a common lake.

Other possible measures are color distances, various measures of local contrast, or even motion or texture dissimilarity between adjacent catchment basins.

6 Connectivity and taxonomy classes

The notion of a connected set in E is well defined if E is a topological space. In [6], Serra generalized this concept by the introduction of a connectivity class. Connectivity classes define the subsets of E which are connected. Hence they help decomposing every set $X \in \mathcal{P}(E)$ into its connected components. Connectivity classes have been extensively studied by Serra and Ronse ([8],[4]).

We extend here the notion of connectivity classes and define taxonomy classes. We present in parallel the notions related to partitions and to hierarchies. For our presentation of the binary case, we largely follow Henk Heijmans, who gives a clear presentation of the developments linked to connectivity in [2].

6.1 Connectivity and taxonomy classes

6.1.1 General definition

Connectivity classes

Definition 14 *Let E be an arbitrary nonempty set. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a connectivity class if it satisfies*
(C1) $\varnothing \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for $x \in E$
(C2) if $C_i \in \mathcal{C}$ and $\bigcap\limits_{i \in I} C_i \neq \varnothing$, then $\bigcup\limits_{i \in I} C_i \in \mathcal{C}$

Alternatively, we say that $\mathcal{C}$ defines a *connectivity* on E. An element of $\mathcal{C}$ is called a *connected set*. This definition is "generative" : larger connected sets are generated from elementary ones with a non empty intersection.

Taxonomy classes

Definition 15 *Let E be an arbitrary nonempty set. $\mathcal{H} = (\mathcal{H}_i)_{i \in I}$ is called a taxonomy class if $\mathcal{H}_i \subseteq \mathcal{P}(E)$ satisfy*
$\sim\sim\sim$(H1) for $i \in I : \varnothing \in \mathcal{H}_i$ and $\{x\} \in \mathcal{H}_i$ for $x \in E$
(H2) $\mathcal{H}_i \subset \mathcal{H}_j$ for $j > i$
(H3) if for $i \in J \subset I, k \in$: $H^k \in \mathcal{H}_i$ and $\bigcap\limits_{k \in K} H^k \neq \varnothing$, then $\bigcup\limits_{k \in K} H^k \in \mathcal{H}_{\max(J)}$

$\mathcal{H}$ is called a taxonomy, each $\mathcal{H}_i$ a taxonomy class and $H^k \in \mathcal{H}_i$ a taxon of level i.

This definition is compatible with the definition of a connectivity class, in the case where $\mathcal{H}$ contains only one element : if J contains only one index l, then $\mathcal{H}_l$ is a connectivity class, as the axioms (C1) and (C2) are verified. This shows that any taxonomy class is a series of nested connectivity classes. Inversely, it is obvious that a series of nested connectivity classes is a taxonomy.

6.1.2 Adjacency relations

Connectivity classes

An important subclass of connectivity classes is based on adjacency.

Definition 16 *A binary relation $\sim$ on $E \times E$ is called an adjacency relation if it is reflexive ($x \sim x$ for every x) and symmetric ($x \sim y$ iff $y \sim x$).*

Given an adjacency relation on $E \times E$, we call $x_0, x_1, ..., x_n$ a *path* between $x = x_0 \sim x \sim .... \sim \ x_n = y$. Define $\mathcal{C}_\sim \subseteq \mathcal{P}(E)$ as the collection of all $C \in E$ such that any two points in C can be connected by a path that lies entirely in C.

Proposition 17 *If $\sim$ is an adjacency relation on $E \times E$, then $\mathcal{C}_\sim$ is a connectivity class.*

Proof. (C1) is obvious. If $C_i \in \mathcal{C}_\sim$ and $z \in \bigcap\limits_{i \in I} C_i$, we have to show that any two points x, y in $\bigcup\limits_{i \in I} C_i$ can be connected by a path that lies entirely in $\bigcup\limits_{i \in I} C_i$. There exists two indices in I such that $x \in C_{i_1}$ and $x \in C_{i_2}$. There exists a path linking x with z in C_{i_1} and a path linking z with y in C_{i_2}. The path between x and y is obtained by concatenating both paths. ∎

Definition 18 *$\mathcal{C}$ is a strong connectivity class if there exists an adjacency relation $\sim$ on $E \times E$ such that $\mathcal{C}^{\sim}$ and E is connected. We say that E possesses a strong connectivity.*

Taxonomy classes

For defining a taxonomy class we need a series of nested adjacency relations (the adjacency between a cat and another cat, or between a cat and a tiger, or a tiger and a mammal cannot be the same).

Definition 19 *A family $(\widetilde{i})_{i \in I}$ of adjacency relations is nested if $x \ \widetilde{i} \ y$ implies $x \ \widetilde{j} \ y$ for $j > i$.*

To each adjacency relation $\widetilde{i}$ we associate its connectivity class $\mathcal{C}_{\widetilde{i}}$

Proposition 20 *If the family $(\widetilde{i})_{i \in I}$ of adjacency relations is nested, then the family $\mathcal{H} = (\mathcal{C}_{\widetilde{i}})_{i \in I}$ is a taxonomy class.*

Proof. (H1) is trivially verified. (H2) is verified as $x \ \widetilde{i} \ y$ implies $x \ \widetilde{j} \ y$ for $j > i$, hence $\mathcal{C}_{\widetilde{i}} \subset \mathcal{C}_{\widetilde{j}}$. Let us prove (H3). Suppose that for $i \in J : H^k \in \mathcal{C}_{\widetilde{i}}$ and $\bigcap\limits_{k \in K} H^k \neq \varnothing$. If $l = \max(J)$ is the maximal index of J, we have $H^k \in \mathcal{C}_{\widetilde{i}} \subset \mathcal{C}_{\widetilde{l}}$. And as $\mathcal{C}_{\widetilde{l}}$ is a connectivity class, $\bigcap\limits_{k \in K} H^k \neq \varnothing$ implies $\bigcup\limits_{k \in K} H^k \in \mathcal{C}_{\widetilde{l}}$. ∎

Example

Consider a grey tone image f defined on a grid with a neighborhood relation. We define the adjacency relation $p \ \widetilde{i} \ q$ by the following conditions:

- p and q neighbors on the grid
- $|f_p - f_q| \leq i$

For the value i, the connected components are the lambda-flat zones of slope i. For increasing values of i, this slope increases and so do the lambda flat zones.

6.1.3 Connectivity openings

Connectivity classes

Serra in [6] has shown that any connected class $\mathcal{C}$ is equivalent with the datum of a connected opening, defined through its invariance domain. If $\mathcal{C}_x$ denotes the subclass of $C \in \mathcal{C}$ that contains a given point ,

$$\mathcal{C}_x = \{C : x \in C \subset \mathcal{C}\}$$

then the union of each non-empty family of sets of $\mathcal{C}_x$, all containing x still belongs to $\mathcal{C}_x$, because of (C2). Hence $\mathrm{Inv}(\gamma_x) = \mathcal{C}_x \cup \{\varnothing\}$ is the invariant set of an opening γ_x, called connected opening of origin x. Its expression is

$$\gamma_x(X) = \bigcup \{C : x \in C \subset \mathcal{C} \text{ and } C \subseteq X\}$$

Since any $x \in E$ belongs to a connected set of $\mathcal{C}$, we have

$$\mathcal{C} = \bigcup_{x \in E} \mathrm{Inv}(\gamma_x)$$

Proposition 21 *Assume that $\mathcal{C}$ is a connectivity on E, then the following conditions are satisfied:*
(O1) every γ_x is an opening
(O2) $\gamma_x(\{x\}) = \{x\}$
(O3) $\gamma_x(X) \cap \gamma_y(X) = \varnothing$ or $\gamma_x(X) = \gamma_y(X)$
(O4) $x \notin X \Rightarrow \gamma_x(X) = \varnothing$
Conversely if γ_x, $x \in E$, is a family of operators satisfying (O1)-(O4) then $\mathcal{C} = \bigcup_{x \in E} \mathrm{Inv}(\gamma_x)$ defines a connectivity.

The principal interest of connection openings lies in the following corollary of [6]

Corollary 22 *Openings γ_x partition any $X \subseteq E$ into the smallest possible number of components belonging to the class $\mathcal{C}$.*

Given a set $X \subseteq E$, every connected component $\gamma_x(X)$ of X is called a *grain* of X. The next result ([2]) says that every connected subset of X is contained within some grain of X

Proposition 23 *Given a connectivity on E and a set $X \subseteq E$. If $C \subseteq X$ is a connected set, then C is contained within some grain of X.*

Another useful property ([6]), shows that x plays no particular role in $\gamma_x(X)$.

Corollary 24 *For all $x, y \in E$ and all $X \subseteq E$ we have*
$y \in \gamma_x(X) \Leftrightarrow \gamma_x(X) = \gamma_x(X)$ and in particular $y \in \gamma_x(X) \Leftrightarrow x \in \gamma_y(X)$

And finally the link between connective classes and partitions.

Definition 25 *Given a space E, a function $P : E \to P(E)$ is called a partition of E if*
(i) $x \in P(x)$, $x \in E$
(ii) $P(x) = P(y)$ or $P(x) \cap P(y) = \varnothing$, for $x, y \in E$

If E is endowed with a connectivity $\mathcal{C}$ and if $P(x) \in \mathcal{C}$ for every $x \in E$, then we say that the partition P is connected.

Given a connective class, every binary image (i.e.set) $X \subseteq E$ can be associated with a connected partition $P(X)$ where the zones of $P(X)$ are the grains of X and X^c. The zone of $P(X)$ containing a point p is :

$$P(X)(p) = \begin{array}{l} \gamma_p(X) \text{ if } p \in X \\ \gamma_p(X^c) \text{ if } p \notin X \end{array}$$

Corollary 26 *For all $x, y \in E$ and all $X \subseteq E$ we have*
$y \in P(X)(x) \Leftrightarrow P(X)(x) = P(X)(y)$ and in particular $y \in P(X)(x) \Leftrightarrow x \in P(X)(y)$

Proof. If $x \in X$, $y \in P(X)(x) = \gamma_x(X) \Rightarrow P(X)(x) = \gamma_x(X) = \gamma_y(X) = P(X)(y)$ and $x \in \gamma_y(X) = P(X)(y)$
If $x \in X^c$, the proof is similar, replacing X by X^c ■

Corollary 27 *For all $x, y \in E$ and all $X \subseteq E$ we have*
$y \notin P(X)(x) \Leftrightarrow P(X)(x) \cap P(X)(y) = \varnothing$

Proof. If $x \in X$ and $y \notin X$, or vice-versa, then the implication is obvious. Consider the case where x, y both belong to X or both belong to X^c. Suppose that there exists a point $z \in P(X)(x) \cap P(X)(y)$; this would imply that $P(X)(x) = P(X)(z) = P(X)(y)$ which contradicts the hypothesis ■

Connected operators

Definition 28 *An operator ψ on $\mathcal{P}(E)$ is connected if the partition $P(\psi(X))$ is coarser than $P(X)$ for every set $X \subset E$*

Taxonomy classes

Consider a taxonomy class $\mathcal{H} = (\mathcal{H}_i)_{i \in I}$. Each $\mathcal{H}_i$ is a connectivity class to which is associated a connection opening γ_x^i.

$\mathcal{H}_i$ also segments every binary image (i.e.set) $X \subseteq E$ into a connected partition $P^i(X)$. The grains of this partition are the sets $P^i(X)(x)$, for $x \in E$.

Lemma 29 *For $j > i$, we have $P^i(X)(x) \subset P^i(X)(x)$ for $x \in E$*

Proof. $\gamma_x^i(X) = \bigcup\{C : x \in C \subset \mathcal{H}_i \text{ and } C \subseteq X\}$ and $\mathcal{H}_i \subset \mathcal{H}_j$ for $j > i$, it follows that $\gamma_x^i(X) \subseteq \gamma_x^j(X)$ and $P^i(X)(x) \subset P^j(X)(x)$ for $j > i$. ■

Proposition 30 *The family $(P^i(X)(x))_{i \in I, x \in E} \cup \{\varnothing\}$ forms a hierarchy. We call it $\mathcal{X}_{\mathcal{H}}(X)$ and write $\chi_{\mathcal{H}}(X)$ for its associated ultrametric half-distance*

Proof. We have to verify that both the intersection axiom and union axiom are satisfied
(Intersection axiom)
Consider two sets of the hierarchy $P^i(X)(q)$ and $P^j(X)(p)$ for $i \leq j$
a) $q \in P^j(X)(p)$: then $P^j(X)(p) \subset P^j(X)(q)$ and since $P^i(X)(q) \subset P^j(X)(q)$, the result $P^i(X)(q) \subset P^j(X)(p)$ is proved
b) $q \notin P^j(X)(p)$ then $P^j(X)(p) \cap P^j(X)(q) = \varnothing$. Since $P^i(X)(q) \subset P^j(X)(q)$ we also have $P^i(X)(p) \cap P^j(X)(q) = \varnothing$
(union axiom)
Consider a set $A = P^j(X)(p)$ of the hierarchy. For a point $q \in P^j(X)(p)$ we have also $P^j(X)(q) = P^j(X)(p)$. Since $P^i(X)(q) \subset P^j(X)(q)$ for $i \leq j$, it shows that $P^j(X)(q) \subset P^j(X)(p)$
Hence $P^j(X)(p) = \bigcup\limits_{q \in A, i < j} P^i(X)(q)$ is indeed the union of the elements of the hierarchy it contains. ■

The family $(P^i(X)(x))_{i \in I, x \in E}$ forming a hierarchy, we may apply to it all results we have established in the first part of the paper, in particular we may associate to it an ultrametric half-distance $\chi(X)$

Connected operators

Definition 31 *An operator ψ on $\mathcal{P}(E)$ is connected if the hierarchy $\mathcal{X}_{\mathcal{H}}(\psi(X))$ is coarser than $\mathcal{X}_{\mathcal{H}}(X)$ for every set $X \subset E$*

This means that the identity operator from the half-metric space E with the half distance $\chi_{\mathcal{H}}(X)$ into the half-metric space E with the half distance $\chi_{\mathcal{H}}(\psi(X))$ is Lipschitz, as for any two points $p, q \in E$, we have $\chi_{\mathcal{H}}(\psi(X))(p, q) \leq \chi_{\mathcal{H}}(X)(p, q)$

7 Conclusion

We may now give a summary of the results, which are all linked to the properties of the ultrametric half distances.:

- each point is centre of a ball
- two balls with the same radius are either disjoint or identical
- two balls with a different radius are either disjoint or one is included in the other

Finally, rather than starting with a hierarchy and defining an ultrametric half-distance, we rather start with an ultrametric half-distance which helps deriving all other concepts:

- taxon at level i: subset $A \in \mathcal{P}(E)$ with a diameter equal to i or empty set

- two taxons A and B have an empty intersection or they form a taxon with a diameter equal to $\mathrm{diam}(A) \vee \mathrm{diam}(B)$
- the connected opening : $\gamma_p^i(X) = \gamma_x^i(X) = \bigcup \{C : x \in C \subset \mathcal{H}_i \text{ and } C \subseteq X\}$
- The grain at level i of the hierarchy associated to X : $P^i(X)(p) =$
  $\gamma_p^i(X)$ if $p \in X$
  $\gamma_p^i(X^c)$ if $p \notin X$
- Adjacence relations : the ultrametric half distance χ may be defined if one knows the distance $\chi(p,q)$ for a collection of pairs of points (p,q). From them it is possible to construct all taxonomy classes and obtain the hierarchy by adding the singletons and the empty set.

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